An Iterative Procedure for Determining Limit Cycles Using Lagrangian Mechanics

John E. Prussing*
University of Illinois, Urbana, Ill.
and
Jack C. Wiley†
Deere and Company, Moline, Ill.

In many nonlinear autonomous mechanical systems, determination of the amplitudes and periods of limit cycles by direct numerical integration of the system differential equations can be very time-consuming computationally. In this investigation, an efficient iterative algorithm which converges to limit cycles of single-degree-of-freedom systems is presented. It is based on the work done by nonconservative forces in the system. Numerical results for several illustrative examples are given, including the van der Pol equation, a feedback control system with hysteresis, a system having an infinite number of stable and unstable limit cycles, and an oscillator with nonlinear dry friction.

I. Introduction

THE limit-cycle behavior of autonomous dynamical systems can be investigated either by the classical approximate techniques of nonlinear analysis or by direct numerical integration of the equations of motion, starting from arbitrarily selected initial conditions. Whereas the direct numerical integration approach is appealing in its simplicity and generality, it can be very time-consuming computationally. This is true especially for systems which are lightly damped.

In this paper, an efficient iterative technique for limit-cycle determination is presented and illustrated for single-degree-of-freedom systems. The technique employs an energy rather than geometrical concept in describing limit cycles, namely, that the net work done by the nonconservative forces is zero over one traversal of the limit cycle. An algorithm is derived for single-degree-of-freedom systems, but in a framework which is amenable to extension to systems of higher order.

Other algorithms exist[1-3] which iteratively locate limit cycles of dynamic systems, but they are less advantageous when applied to single-degree-of-freedom systems. The advantages of the new technique are that 1) iteration is performed on a scalar function rather than a two-component state vector and the limit cycle period; 2), the initial estimate of the limit cycle period is determined automatically by algorithm; and 3) the iteration never will converge to an unstable singular point (as was the case in Ref. 3) but, by a simple artifice, can be made to converge to an unstable limit cycle, if desired. The technique is based on a Newton iteration on the work done by the nonconservative forces and is derived using the concepts of Lagrangian mechanics.

II. Problem Description

Consider a holonomic, scleronomic, nondegenerate single-degree-of-freedom mechanical system with configuration specified by the generalized coordinate $q(t)$. For this type of system, the kinetic energy T is a homogeneous quadratic function of the generalized velocity $\dot{q}$[4,5]:

$$T=\tfrac{1}{2}\, m(q)\, \dot{q}^2 \tag{1}$$

where $m(q)$ is the (positive) generalized mass of the system.

If no active forces are explicitly time-dependent, and if the potential energy does not depend explicitly on the velocities, the potential energy V of all of the conservative forces and the nonconservative generalized force Q have the functional forms

$$V=V(q),\ Q=Q(q,\dot{q}) \tag{2}$$

Lagrange's equation of motion for this system yields

$$m(q)\ddot{q}+\tfrac{1}{2}m'(q)\dot{q}^2+V'(q)=Q(q,\dot{q}) \tag{3}$$

where the prime denotes differentiation with respect to the argument q.

To describe the equation of motion in a state-space formulation, define the two-dimensional state vector $\boldsymbol{x}(t)$: $\boldsymbol{x}^T(t)=[x_1(t)x_2(t)]^T=[q(t)\dot{q}(t)]^T$. The equation motion Eq. (3) then can be written in first-order form as

$$\dot{x}_1=x_2 \tag{4a}$$

$$\dot{x}_2=[Q(x_1,x_2)-\tfrac{1}{2}m'(x_1)x_2^2-V'(x_1)]/m(x_1) \tag{4b}$$

The analysis which follows considers systems of the form of Eq. (4) which possess at least one stable or unstable limit cycle, i.e., an isolated periodic solution $\boldsymbol{x}^*[t;\boldsymbol{x}^*(t_0)]$. The limit cycle $\boldsymbol{x}^*(t)$ is an asymptotic periodic solution to which the state $\boldsymbol{x}(t)$ converges at $t\to+\infty$ (or, in the unstable case, as $t\to-\infty$) from any initial state $\boldsymbol{x}(t_0)$ in a certain region of the state space.

The criterion used to determine a state $\boldsymbol{x}^*(t_0)$ on a limit cycle of the system is that the value of a certain functional must be zero on a limit cycle. This functional is the work done by the nonconservative forces $Q(x_1,x_2)$ during one complete traversal of the limit cycle. Under certain restrictions, which are derived for a single-degree-of-freedom system, a zero value of this functional is a sufficient, as well as necessary, condition for the existence of a limit-cycle solution.

Analysis

Assume that the system of Eq. (4) possesses at least one limit-cycle solution and that the origin of the x_1-x_2 state

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*Associate Professor, Department of Aeronautical and Astronautical Engineering. Member AIAA.
†Senior Research Engineer, Technical Center. Member AIAA.

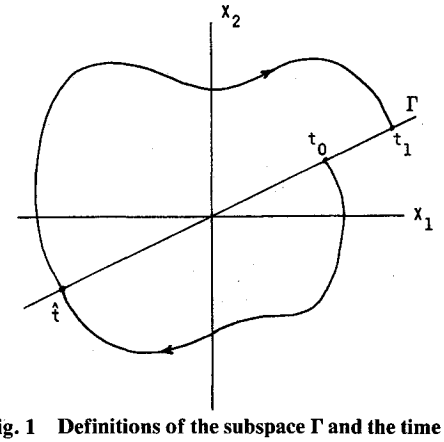

Fig. 1 Definitions of the subspace Γ and the time t_1.

(phase) plane is an interior point of the limit cycle. Since $\dot{x}_1 = x_2$, all singular points of the system lie on the subspace $x_2 = 0$. Thus, by appropriate definition of the generalized coordinate $x_1(=q)$, the origin always can be made an interior point of the limit cycle. For example, in systems having a single unstable singular point, x_1 is defined in such a way that the singular point is at the origin of the state plane.

In order to calculate the work done by the nonconservative generalized force $Q(x_1,x_2)$ during one complete encirclement of the origin of the state plane, one must define precisely a complete encirclement from an arbitrary initial state $x(t_0)$. Consider the one-dimensional subspace Γ which is the space of all states spanned by the initial state:

$$\Gamma = \{x : x = \lambda x(t_0); -\infty < \lambda < \infty\} \tag{5}$$

Furthermore, as shown in Fig. 1, define t_1 as the subsequent time at which the state next lies in the subspace Γ in the same half-plane as the initial state:

$$x(t_1) = \lambda x(t_0),\ \lambda > 0 \tag{6}$$

The work done by the nonconservative generalized force $Q(x_1,x_2)$ during the time interval $t_0 < t < t_1$ is

$$W(t_1,t_0) = \int_{x_1(t_0)}^{x_1(t_1)} Q(x_1,x_2)\,dx_1 \tag{7}$$

where the integral is evaluated over the path $x[t;x(t_0)]$.

As shown in Appendix A, under fairly weak restrictions, a necessary and sufficient condition that $x(t_0)$ is a state $x^*(t_0)$ on a limit cycle of the system is that $W(t_1,t_0) = 0$. The restrictions are that

$$x_1 m'(x_1) \geq -2m(x_1) \tag{8a}$$

$$x_1 V'(x_1) \geq 0 \tag{8b}$$

namely, that $m(x_1)$ not decrease too rapidly with increasing $|x_1|$ and that $V(x_1)$ not decrease with increasing $|x_1|$. Note that no assumptions are required on the functional form of $Q(x_1,x_2)$. Thus, Eq. (8b) always can be satisfied, if necessary, by incorporating all of the conservative forces into $Q(x_1,x_2)$, resulting in $V(x_1) = 0$. The restriction of the generalized mass $m(x_1)$ in Eq. (8a) is fairly weak; e.g., a constant $m(x_1)$ satisfies the condition, since $m(x_1) > 0$.

Since the system is autonomous, the value of the initial time t_0 can be taken to be zero. The time required for one complete encirclement of the origin is then t_1. The iteration process generates a sequence of estimates of the initial state $x(0)$: $\{x^{(0)}(0), x^{(1)}(0),\ldots, x^{(k)}(0)\ldots\}$ which converges to a state on the limit cycle $x^*(0)$. On the kth iteration, $x^{(k)}$ is the current estimate of a state on the limit cycle, and $t_1^{(k)}$ is the current estimate of the limit-cycle period. Once the iteration process has converged, $x[t;x^*(t_0)]$ is a limit cycle of the system.

The functional $W(t_1,0)$, which, on a given iteration, is actually $W^{(k)}(t_1^{(k)}, 0)$, can be evaluated without explicit integration as implied by the definition in Eq. (7). As shown in Appendix B, for an autonomous system, the work done by the nonconservative forces can be expressed in terms of the Hamiltonian function H as

$$W(t_1,0) = H(t_1) - H(0) \tag{9}$$

In addition, since the kinetic energy is a homogeneous quadratic function of the generalized velocity $\dot{q}$ [Eq. (1)], H is simply the total mechanical energy of the system $H = T + V$. Thus, on each iteration,

$$W(t_1,0) = \tfrac{1}{2}\{m[x_1(t_1)]x_2^2(t_1) - m[x_1(0)]x_2^2(0)\} + V[x_1(t_1)] - V[x_1(0)] \tag{10}$$

Equation (10) is the form for the work done by the nonconservative forces used in the algorithm.

III. Iteration Algorithm

Once the expression for $W(t_1,0)$ has been derived for a particular system, an algorithm must be implemented to iterate on the initial condition $x(0)$ in the subspace Γ and converge to a value $x^*(0)$ on the limit cycle. The method developed in this section is a modified Newton iteration on the scalar $W(t_1,0)$. A different iteration technique could be used; the specific technique employed is not a major concept on the overall algorithm.

To formulate the iteration algorithm, let the superscript k denote the iteration index, $k = 0,1,2,\ldots$, and consider the first-order predicted change in $W^{(k)}(t_1^{(k)},0)$ due to a small change in the initial condition, $\delta x^{(k)}(0)$. Abbreviating $W^{(k)}(t_1,{}^{(k)}0)$ as $W^{(k)}$, the first-order predicted change in the work done is

$$\delta W^{(k)} = g^{T(k)}\,\delta x^{(k)}(0) \tag{11}$$

where

$$g^{T(k)} \triangleq \partial W^{(k)}/\partial x^{(k)}(0) \tag{12}$$

In order that the next iterate of the initial state $x^{(k+1)}(0)$ lie in the subspace Γ, it is necessary that

$$\delta x^{(k)} = \mu^{(k)} x^{(k)}(0) \tag{13}$$

for some scalar $\mu^{(k)}$. The iteration algorithm is obtained by solving for the value of $\delta x^{(k)}(0)$ which yields zero for the first-order predicted value of $W^{(k+1)}$. To accomplish this, set

$$\delta W^{(k)} = -W^{(k)} \tag{14}$$

Combining Eqs. (11, 13, and 14), the iteration algorithm becomes

$$\delta x^{(k)}(0) = [-W^{(k)}/g^{T(k)}x^{(k)}(0)]\,x^{(k)}(0) \tag{15}$$

The initial condition then is incremented by $x^{(k+1)}(0) = x^{(k)}(0) + \epsilon\,\delta x^{(k)}(0)$; $0 < \epsilon \leq 1$. The factor ϵ is included to limit

Table 1 Results for the van der Pol system

k	$x_1^{(k)}(0)$	$x_1^{(k)}(\tau)$	$P^{(k)}$	$W^{(k)}$
0	11.000	−3.1144	16.481	-5.5650×10^{1}
1	3.1144 [a]	−2.6343	6.4678	-1.3800×10^{0}
2	2.4269	−2.2845	6.3299	-3.3561×10^{-1}
3	2.1063	−2.0758	6.2951	-6.3710×10^{-2}
4	2.0093	−2.0068	6.2878	-5.1484×10^{-3}
5	2.0000	−2.0000	6.2871	-5.9744×10^{-5}

[a] = natural estimate used.

the size of the iteration step if $|\delta x^{(k)}(0)|$ is so large that it violates grossly the first-order approximation of the algorithm. To determine the value of ϵ, one specifies that $|\delta x(0)|/|x(0)| \leq m$, where m is a specified maximum permissible relative change in the initial state. If $|\delta x(0)|/|x(0)| < m$, $\epsilon = 1$. Otherwise, $\epsilon = m\ |x(0)|/|\delta x(0)|$. This limitation on the iteration step size was necessary in the last example system of Sec. IV.

Since $W^{(k)}(t_1,0) = H^{(k)}(t_1) - H^{(k)}(0)$ [Eq. (9)], the gradient vector has the form

$$g^{T(k)} = \frac{\partial W^{(k)}(t_1,0)}{\partial x^{(k)}(0)} = \frac{\partial H^{(k)}(t_1)}{\partial x^{(k)}(t_1)} \frac{\partial x^{(k)}(t_1)}{\partial x^{(k)}(0)} - \frac{\partial H^{(k)}(0)}{\partial x^{(k)}(0)} \tag{16}$$

In Eq. (16), the partial derivatives of H are determined analytically from the expression for H. The matrix $\partial x^{(k)}(t_1)/\partial x^{(k)}(0)$ is the 2×2 state-transition matrix[6] $\Phi^{(k)}(t,0)$ for the variational equation to the system $\dot{x} = f(x)$, evaluated at $t = t_1$. [$f(x)$ is the right-hand side of Eq. (4)]. This state-transition matrix satisfies the differential equation:

$$(d/dt)\Phi^{(k)}(t,0) = F^{(k)}(t)\Phi^{(k)}(t,0); \Phi^{(k)}(0,0) = I \tag{17}$$

where $F^{(k)}(t)$ is the Jacobian matrix $\partial f/\partial x$ evaluated along the solution to $\dot{x} = f(x)$, for the initial condition $x^{(k)}(0)$, and I is the 2×2 identity matrix.

The choice of the subspace $x_2 = 0$ for Γ can offer both conceptual and computational advantages. Conceptually, the algorithm iterates on $x_1(0)$, which is the maximum excursion of the generalized coordinate q. The value of $x_1(0)$ required to start the iteration process may be estimated more readily compared with using an arbitrary subspace Γ. In addition, for the hysteresis control system example of Sec. IV, the value of the maximum excursion is required for use as a parameter in the differential equation of the system.

Computationally, using $x_2 = 0$ as Γ requires computation of only the first column of the transition matrix $\Phi^{(k)}(t_1,0)$. Since $x_2 = 0$, $g^T x(0) = g_1 x_1(0)$, Eq. (15) becomes

$$\delta x^{(k)}(0) = -\frac{W^{(k)}}{g_1^{(k)}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{18}$$

where g_1 is the first component of the gradient vector of Eq. (16).

Symmetric Limit Cycles

A major computational savings is possible in those systems which have limit cycles which are symmetric about the origin of the phase plane; i.e., if a state x^* is on the limit cycle, $-x^*$ is also. In this case, $x^*[t + (P/2); x^*(0)] = -x^*[t; x^*(0)]$, where P is the limit-cycle period. The computational savings results from the fact that the numerical integration can be terminated after a half-encirclement of the origin of the phase plane. The variable t_1 can be replaced by $\hat{t}$ (Fig. 1) in the algorithm, since $W[P/2,0] = W[P,0] = 0$ [Eq. (10)]. As the iteration converges to the limit cycle, $\hat{t} \to P/2$.

As shown in Appendix C, for single-degree-of-freedom systems a sufficient condition for the limit cycle to be symmetric about the origin is that $f(x)$ is an odd vector-valued function of the state vector x. In systems for which $f(x)$ has the form of Eq. (4), f is odd if 1), m and V are even functions of x_1, and 2) $Q(x_1,x_2)$ is an odd function of its pair of arguments. This property occurs in many systems, including all but the last example system presented in Sec. IV.

A Modified Algorithm

In this section, a modified algorithm is described which accelerates convergence to the limit cycle. In addition, it insures that the process converges to a stable limit cycle (or unstable, if the user so chooses). Consider a single-degree-of-freedom system having a stable limit cycle. For an initial condition $x^{(k)}(0)$ lying in a certain domain outside the limit cycle in the phase plane, numerical integration of the system differential equation yields a trajectory which spirals inward, asymptotically approaching the limit cycle. Analogously, for an initial condition in a certain domain inside the limit cycle, the trajectory spirals outward.

After integration from an initial state $x^{(k)}(0)$ to the time t_1 (or $\hat{t}$ for the symmetric case), $x^{(k)}(t_1)$ may be interpreted as a "natural" estimate of a state on the limit cycle. If no iterative algorithm were used, the integration would continue using $x^{(k)}(t_1)$ as the initial condition for the succeeding integration. By monitoring this "natural" estimate one can determine whether or not the algorithm estimate is better than the natural estimate; the better estimate then is used for the next iteration.

Consider the case $|x^{(k)}(t_1)| < |x^{(k)}(0)|$, symptomatic of an inward-spiraling trajectory. If the algorithm estimate generated is larger than the natural estimate in magnitude, i.e., if $|x^{(k+1)}(0)| > |x^{(k)}(t_1)|$, the natural estimate clearly is better than the algorithm estimate (nearer to the limit cycle) and is the preferred estimate to use for the next iteration. For a trajectory spiraling outward in the phase plane, $|x^{(k+1)}(0)| < |x^{(k)}(t_1)|$ is an indication that the natural estimate should be used.

This observation suggests modifying the algorithm so that the algorithm estimate is replaced by the natural estimate whenever the natural estimate is superior. As shown in the numerical results of Sec. IV, the modified algorithm provided useful in moderate-to-heavily-damped systems in which the initial estimate was relatively far from the limit cycle. Clearly, the convergence rate of the modified algorithm, at worst, is equal to the rate for direct numerical integration. In all cases studied, the rate was considerably better.

One further observation can be made concerning the modified algorithm. When integrating forward in time, the modified algorithm cannot converge to an unstable singularity or an unstable limit cycle, since the natural estimate proceeds away from the unstable singularity or limit cycle. A convergence to an unstable singularity was reported for the algorithm of Ref. 3. If, however, one wishes to converge to an unstable limit cycle, this is accomplished by integration backwards in time; the algorithm then converges to the unstable limit cycle. An illustration of this is given in Sec. IV.

Summary of the Algorithm

Considering $x_2 = 0$ as the subspace Γ, for a given initial estimate of the limit-cycle amplitude $x_1^{(k)}(0)$ $(k = 0)$, the algorithm proceeds as follows:

1) Numerically integrate the system differential equations (4) from initial conditions $x_1(0) = x_1^{(k)}(0)$, $x_2(0) = 0$, along with the differential equation for the first column of the state transition matrix (17) for $0 \leq t \leq t_1$.

2) Calculate the work done $W(t_1,0)$ by the nonconservative forces using Eq. (10).

3) Calculate the gradient g_1 of the work with respect to $x_1(0)$ using Eq. (16).

4) Calculate the required change in the initial state $\delta x^{(k)}(0)$ using Eq. (18). Increment the current estimate $x^{(k)}(0)$ according to $x^{(k+1)}(0) = x^{(k)}(0) + \epsilon \delta x^{(k)}(0)$ $(0 < \epsilon \leq 1)$, where ϵ is chosen as described preceding Eq. (16).

5) Check whether the natural estimate $x^{(k)}(t_1)$, as discussed in the modified algorithm segment of Sec. III, is superior to the algorithm estimate. If it is, replace $x^{(k+1)}(0)$ by $x^{(k)}(t_1)$ for use in the next iteration.

6) Redo steps 1-5 until the value of $|W(t_1,0)|$ is sufficiently small, or until $|x^{(k)}(t_1) - x^{(k)}(0)|$ is sufficiently small.

IV. Numerical Results

The iteration algorithm was applied successfully to several second-order systems. Rapid convergence to a limit cycle was obtained for four relatively simple systems and three more complicated examples. Details of the iterations are given for three of these systems. The numerical intergration algorithm used was the Bulirsch-Stoer extrapolation method. The four simple systems studied are of the form $\ddot{x}+h(x,\dot{x})=0$, where

$$h(x,\dot{x})=\begin{cases} -0.1\dot{x}(1-x^2)+x & (19)^9 \\ -0.1\dot{x}(1-1/3\dot{x}^2)-x(1-x^2) & (20)^7 \\ -0.1\dot{x}(1-\dot{x}^2)+x & (21)^2 \\ +0.1\dot{x}(\dot{x}^2+x^2-a^2)+x & (22)^8 \end{cases}$$

Each of the preceding systems, although represented by a differential equation rather than the mechanical quantities T, V, and Q is cast easily into the form of Eq. (4). For example, Eq. (19) can be represented by $m(x_1)\equiv 1$, $V(x_1)=\frac{1}{2}x_1^2$, $Q(x_1,x_2)=0.1\ x_2(1-x_1^2)$. It is evident that this system satisfies the conditions for a symmetric limit cycle, given in Sec. III.

The numerical results for the van der Pol system (19) are given in Table 1. The subspace $x_2=0$ was used as Γ; thus, each iteration is represented by the value of $x_1(0)$. Since the limit cycle is symmetric, $\hat{\tau}$ is used in place of t_1. The symbol # following the value of $x_1(0)$ denotes that the natural estimate was used in place of the algorithm estimate on that iteration. The values of the amplitude and period obtained for the limit cycle agree with those of Ref. 9. The behavior of the algorithm applied to the other three simple systems was quite similar.

The first of the more complicated systems studied is discussed by Stoker[10]:

$$\ddot{x}-0.1\sin\dot{x}+x=0 \qquad (23)$$

This system is interesting because it possesses an infinite number of (symmetric) limit cycles, alternately stable and unstable. The results of applying the algorithm to this system for two distinct initial conditions are shown in Table 2. Of special note is the limit cycle of amplitude 7.0017. This limit cycle is unstable; convergence was obtained integrating backwards in time ($\hat{\tau}<0$). The system is represented by $m(x_1)\equiv 1$, $V(x_1)=\frac{1}{2}x_1^2$, and $Q(x_1,x_2)=0.1\sin x_2$. Nowhere in the iterations of Table 2 was the natural estimate superior to the algorithm estimate.

Table 2 Results for two limit cycles of the Stoker system

k	$x_1^{(k)}(0)$	$x_1^{(k)}(\hat{\tau})$	$P^{(k)}$	$W^{(k)}$
0	5.0000	−4.8985	6.2851	-5.024×10^{-1}
1	3.3105	−3.3739	6.2843	$+2.1201\times10^{-1}$
3	3.9186	−3.9085	6.2850	-3.9425×10^{-2}
3	3.8337	−3.8336	6.2850	-3.3491×10^{-4}
4	3.8330	−3.8330	6.2850	$+9.3266\times10^{-8}$
0	8.0000	−7.9278	6.2837	-5.7564×10^{-1}
1	6.6946	−6.7215	6.2838	$+1.7997\times10^{-1}$
2	7.0053	−7.0050	6.2838	-2.2359×10^{-3}
3	7.0017	−7.0017	6.2838	-6.5276×10^{-7}

Table 3 Results for the control system with hysteresis

k	$x_1^{(k)}(0)$	$x_1^{(k)}(\hat{\tau})$	$P^{(k)}$	$W^{(k)}$
0	5.0000	−4.3916	2.1667	−2.8568
1	4.1203	−3.8127	2.2194	−1.2199
2	3.6731	−3.5122	2.2596	-5.775×10^{-3}
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
12	3.1472	−3.1468	2.3278	-1.2396×10^{-3}
13	3.1466	−3.1458	2.3280	-2.3725×10^{-3}
14	3.1454	−3.1454	2.3282	-5.9004×10^{-4}

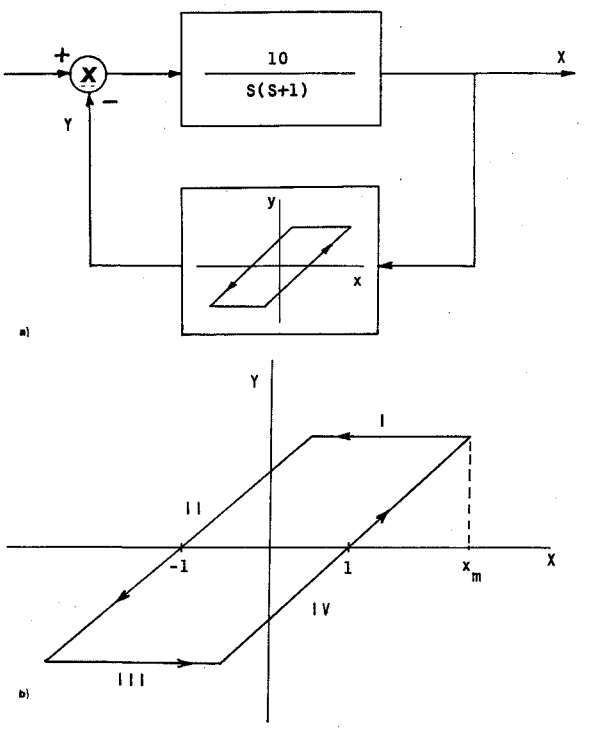

Fig. 2a) Control system with backlash; b) backlash nonlinear element.

A second complicated example is the hysteresis nonlinearity in a feedback system discussed in Ref. 11. A physical example of the system is a positioning control system with backlash in the gears. Figure 2a shows the block diagram of the systems; Fig. 2b illustrates the details of the hysteresis element. The equations of motion for the system are different along each segment of the nonlinear element:

$$\dot{x}_1=x_2 \qquad (24a)$$

$$\dot{x}_2=\begin{cases} -x_2-10(x_m-1) & \text{(I)} \\ -x_2-10(x_1+1) & \text{(II)} \\ -x_2-10(x_m+1) & \text{(III)} \\ -x_2-10(x_1-1) & \text{(IV)} \end{cases} \qquad (24b)$$

In Eqs. (24), x_m is the value of the maximum excursion of the coordinate x_1; this value changes each time x_2 passes through zero. This leads one to choose $x_2=0$ as Γ; then x_m is simply $x_1(0)$ for each iteration. In addition, since the limit cycle is symmetric, only segments (I) and (II) are used in the algorithm. The system can be represented by $m(x_1)\equiv 1$, $V(x_1)\equiv 0$, and $Q(x_1,x_2)=-x_2-10[x_1(0)-1]$ along segment (I), and $-x_2-10(x_1+1)$ along segment (II). The iterations for this system are shown in Table 3.

The last example is from Panovka and Gubanova.[12] As shown in Fig. 3a, the system is a mass-spring oscillator in which the mass slides on a rough, moving belt. The coordinate x is measured from the static equilibrium position of the mass. The constant velocity of the belt is denoted by V_B. The frictional force R is related to the slip velocity of the mass, $v\triangleq V_B-\dot{x}$, by

$$R(\alpha)=R^*\,[\mathrm{sgn}(\alpha)-\alpha+\alpha^3/3]$$

where

$$\alpha\triangleq v/v^*=(V_B-\dot{x})/v^*$$

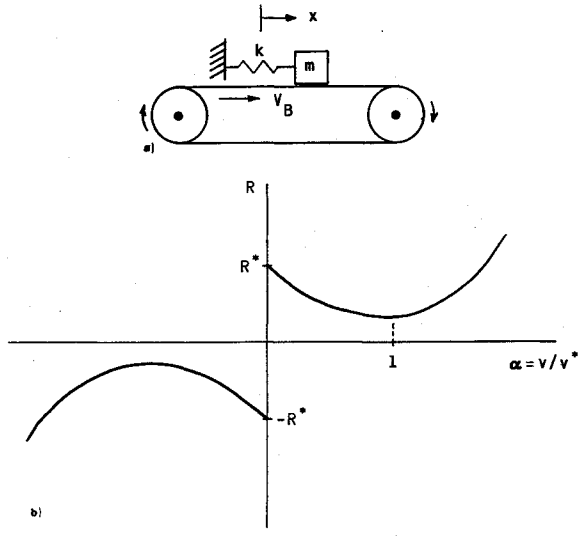

Fig. 3a) Moving belt system; b) friction force characteristic.

The quantities v^* and R^* are shown in Fig. 3b. The equation of motion of the system is

$$m\ddot{x}+k\,x=R(\alpha)-R(\alpha_0)$$

where $\alpha_0 \triangleq V_B/v^*$. For a unit mass, this system can be represented by $m(x_1)\equiv 1$, $V(x_1)=\frac{1}{2}kx_1^2$, $Q(x_1,x_2)=R(\alpha)-R(\alpha_0)$, with $\alpha=(V_B-x_2)/v^*$.

For this system the limit cycle is not symmetric about the origin. An unstable singularity exists at the origin for $\alpha_0<1$ and $4k-(R^*/v^*)^2\,(1-\alpha_0{}^2)^2>0$. The values of the system parameters used are $k=100$, $v^*=2$, $V_B=2^{\frac{1}{2}}$, $R^*=0.3$. The iterations for a convergence to the limit cycle are shown in Table 4. $x_2=0$ was used as the subspace Γ.

Because of the discontinuity in the friction force, this system is very difficult to handle numerically. As shown in Table 4, the iteration step was limited in magnitude on the first two iterations. After the limit cycle was located approximately by iteration, the algorithm chose the natural estimate to converge to the limit cycle. The reader should note that this does not constitute a failure of the algorithm. The algorithm, from an arbitrary initial guess, located a state sufficiently close to the limit cycle that final convergence could be obtained by a relatively small amount of direct integration.

V. Conclusions

The algorithm has proved successful in locating limit cycles for a variety of single-degree-of-freedom systems. In all cases studied, the algorithm converged considerably faster than direct numerical integration. The possibility of extending the algorithm to higher-ordered system, including nonautonomous systems transformed into higher-ordered autonomous systems, is an interesting idea which is being investigated. One potential difficulty appears to be a suitable choice for the subspace Γ.

It is felt that this algorithm, based on the work done by the nonconservative forces in the system, has a certain physical appeal. Although the total energy is not conserved on a limit cycle, the fact that the total work done over one period is zero characterizes a periodic energy balance in the system.

Table 4 Results for the moving belt system

k	$x_1{}^{(k)}(0)$	$x_1{}^{(k)}(t_1)$	$P^{(k)}$	$W^{(k)}$
0	0.30000	0.28863	0.62836	-3.3464×10^{-3}
1	0.20700[a]	0.20043	0.62837	-1.3385×10^{-3}
2	0.14283[a]	0.14347	0.62836	$+9.1283\times10^{-5}$
3	0.14347[#b]	0.14391	0.62837	6.3466×10^{-5}
4	0.14391[#]	0.14411	0.62838	2.8763×10^{-5}
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
10	0.14451[#]	0.14451	0.62837	8.2921×10^{-7}

[a] iteration step limited to 31% of current value.
[b] # = natural estimate used.

Appendix A

To determine the conditions under which a zero value of the work done by the nonconservative forces is a necessary and sufficient condition for a limit cycle, consider the subspace Γ (of the two-dimensional state space) defined by Eq. (5), and the times $t=0$ and t_1 defined by Eq. (6). $x(0)$ is a state on a limit cycle if, and only if, $\lambda=1$. It is convenient to consider separately the general case $x_1(0)\neq 0$ and the special case where Γ is the x_2 axis $[x_1(0)=0, x_2(0)\neq 0]$.

Theorem

For the general case $x_1(0)\neq 0$, subject to conditions 1) $x_1(t)V'[x_1(t)]\geq 0$, 2) $x_1(t)m'[x_1(t)]>-2m[x_1(t)]$, and 3) in the case $V'(x_1)\equiv 0$ on a finite interval, the use of the subspace $x_2=0$ as Γ is excluded, then a necessary and sufficient condition that the state $\boldsymbol{x}(0)$ lies on a limit cycle of the system ($\lambda=1$) is that $W[t_1,0]=0$.

Comments

Conditions 1 and 2 are discussed in the text following Eq. (8). Condition 3 excludes the degenerate case in which a zero value of work done can result regardless of the values of $\boldsymbol{x}(0)$ and $\boldsymbol{x}(t_1)$. Taking the subspace Γ to be $x_2=0$, $W[t_1,0]$ is simply $V[x_1(t_1)]-V[x_1(0)]$. If $V(x_1)$ is constant on a finite interval containing voth $x_1(t_1)$ and $x_1(0)$, $W[t_1,0]$ will be zero even if $x_1(t_1)\neq x_1(0)$.

Proof

The fact that $W[t_1,0]=0$ is a necessary condition for a limit cycle is well known and follows directly from Eq. (10). If $\boldsymbol{x}$ is a state on the limit cycle, $\boldsymbol{x}(t_1)=\boldsymbol{x}(0)$ ($\lambda=1$); hence $W[t_1,0]=0$.

To show that $W[t_1,0]$ is a sufficient condition for a limit cycle subject to the conditions of the theorem, note that, since $x_1(0)\neq 0$, $x_2(0)=kx_1(0)$ for some finite value of k. By the definition of t_1, $x_2(t_1)=kx_1(t_1)$. Using these relationships to eliminate x_2 in Eq. (10),

$$W[t_1,0]=\tfrac{1}{2}k^2\{m[x_1(t_1)]x_1^2(t_1) - m[x_1(0)]x_1^2(0)\}+V[x_1(t_1)] - V[x_1(0)] \tag{A1}$$

In order to prove that $W[t_1,0]=0$ implies that $\lambda=1$, it is convenient to prove the contrapositive statement that $\lambda\neq 1$ implies that $W[t_1,0]\neq 0$.

Assume that $\lambda>1$. Then, based on the assumption on the form of $V[x_1(t)]$, $V[x_1(t_1)]-V[x_1(0)]\geq 0$, since $|x_1(t_1)|>|x_1(0)|$. In addition, the assumption that $x_1(t)m'[x_1(t)]>-2m[x_1(t)]$ implies that

$$x_1^2(t)\{x_1(t)m'[x_1(t)]+2m[x_1(t)]\}>0$$

This is equivalent to

$$x_1(t)\{m[x_1(t)]x_1^2(t)\}'>0$$

for all $x_1(t)\neq 0$. Thus, $m[x_1(t_1)]x_1^2(t_1)-m[x_1(0)]x_1^2(0)>0$ and $W[t_1,0]\geq 0$ [Eq. (A1)] with equality only if $k=0$ and

$V[x_1(t_1)] = V[x_1(0)]$. If one excludes the use of the subspace $x_2 = 0$ ($k = 0$) in the case for which $V'[x_1(t)] \equiv 0$ on a finite x_1 interval, one obtains the desired result that, if $\lambda > 1$, $W[t_1, 0] > 0$.

On the other hand, if one assumes $0 < \lambda < 1$, a similar argument yields $W[t_1, 0] < 0$. Thus, if $W[t_1, 0] = 0$, then $\lambda = 1$ and the initial state $x(0)$ lies on a limit cycle of the system.

Lemma

For the special case $x_1(0) = 0$, $x_2(0) \neq 0$, a necessary and sufficient condition that $x(0)$ lies on a limit cycle of the system is that $W[t_1, 0] = 0$ with no restrictions on the functional form of $m(x_1)$ or $V(x_1)$.

Comment

This corresponds to using the subspace $x_1 = 0$ as Γ. The conditions using this subspace are less restrictive than the general case. However, this subspace is not always a convenient one to use. For example, in the control system with hysteresis treated in Sec. IV, use of the subspace $x_2 = 0$ allows one to estimate directly the maximum excursion (occurring at $x_2 = 0$) for use as a parameter in the differential equation of motion.

Proof

Since $x_1(t_1) = x_1(0) = 0$ for this case,

$$W[t_1, 0] = \tfrac{1}{2}\, m[0]\, [x_2^2(t_1) - x_2^2(0)] \tag{A2}$$

Thus $W[t_1, 0] = 0$ if, and only if, $x_2^2(t_1(0)$ (i.e., $\lambda = \pm 1$). But, by the definition of t_1, $\lambda > 0$. Hence $W[t_1, 0] = 0$ if, and only if, $x_2(t_1) = x_2(0)$.

Appendix B

In this Appendix an expression is derived for the work done by the nonconservative forces in terms of the Hamiltonian function. The derivation is similar to the derivation of Jacobi's intergral.[8]

Consider the Lagrangian function $L = L(q, \dot{q}, t)$:

$$dL/dt = (\partial L/\partial q)\dot{q} + (\partial L/\partial \dot{q})\ddot{q} + (\partial L/\partial t) \tag{B1}$$

From Lagranges equation,

$$\partial L/\partial q = (d/dt)(\partial L/\partial \dot{q}) - Q\ (q, \dot{q}) \tag{B2}$$

where Q is the generalized force acting on the system. Thus

$$dL/dt = (d/dt)[(\partial L/\partial \dot{q})\dot{q}] - Q\,\dot{q} + (\partial L/\partial t) \tag{B3}$$

Defining the Hamiltonian function in the usual manner,

$$H \triangleq (\partial L/\partial \dot{q})\dot{q} - L \tag{B4}$$

$$dH/dt = (d/dt)[(\partial L/\partial \dot{q})\dot{q}] - (dL/dt) \tag{B5}$$

Using Eq. (B3),

$$dH/dt = Q\,\dot{q} - (\partial L/\partial t) \tag{B6}$$

Thus, for a scleronomic system ($\partial L/\partial t \equiv 0$),

$$H(t_1) - H(t_0) = \int_{t_0}^{t_1} Q\,\dot{q}\,dt = \int_{q(t_0)}^{q(t_1)} Q\,dq = W(t_1, t_0) \tag{B7}$$

where $W(t_1, t_0)$ is the work done by the generalized forces $Q(q, \dot{q})$ over the path of the system from $q(t_0)$ to $q(t_1)$. This result is Eq. (9).

Appendix C

Consider the autonomous system $\dot{x} = f(x)$. In this Appendix it is shown that, if $f(x)$ is an odd, vector-valued function of x, then a limit-cycle solution $x^*(t)$ of period P which exists in a two-dimensional state plane has the symmetry property

$$x^*[t + P/2; x^*(0)] = -x^*[t; x^*(0)] \tag{C1}$$

Lemma 1

If $f(x)$ is an odd function of x, then any solution to $\dot{x} = f(x)$ has the property

$$x[t; x(0)] = -x[t; -x(0)] \tag{C2}$$

Proof

Define $z(t) \triangleq -x(t)$. Then $-\dot{z} = f(-z)$. But $f(-z) = -f(z)$ if f is an odd function. Thus $\dot{z} = f(z)$, and $z[t; z(0)] = x[t; x(0)]$. Substituting $z(t) = -x(t)$ yields Eq. (C2).

Lemma 2

If $x^*(t)$ is a limit-cycle solution of period P to the two-dimensional autonomous system $\dot{x} = f(x)$, and if $f(x)$ is an odd function, then

$$x^*[P/2; x^*(0)] = -x^*(0) \tag{C3}$$

Proof

Define the time ξ by

$$x^*[\xi; x^*(0)] = -x^*(0) \tag{C4}$$

To show that such a time ξ exists, consider the following argument. Take any closed path around the origin of the $x_1 - x_2$ plane as shown in Fig. 4. When the state is at point A, the projection of point A across the origin is at A', which is outside the closed path. At sometime later, when the state is at point B, its projection is at B', which is inside the closed path. Since the path is continuous, at some intermediate time between points A and B, there must exist a point C which has its projection C' on the closed path. If one chooses $t = 0$ at point C, then $x^*[\xi; x^*(0)] = -x^*(0)$, where ξ is the time for the state to travel from C to C'.

Using the fact that the system is autonomous and has a limit cycle of period P

$$x^*[P - \xi; x^*[\xi; x^*(0)]] = x^*[P; x^*(0)] = x^*(0) \tag{C5}$$

But, from Eq. (C4),

$$x^*[P - \xi; x^*[\xi; x^*(0)]] = x^*[P - \xi; -x^*(0)] \tag{C6}$$

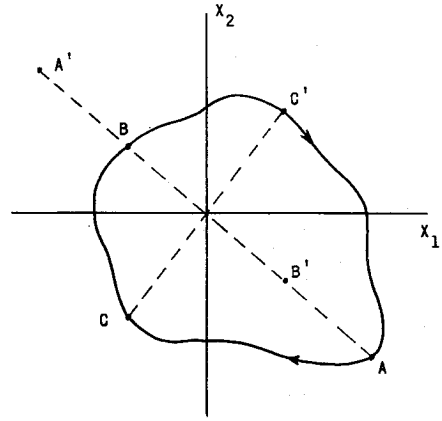

Fig. 4 The existence of a symmetry point.

and, by the property of Lemma 1,

$$x^*[P-\xi; -x^*(0)] = -x^*[P-\xi; x^*(0)] \quad \text{(C7)}$$

Combining Eqs. (C5-C7),

$$x^*[P-\xi; x^*(0)] = -x^*(0) = x^*[\xi; x^*(0)] \quad \text{(C8)}$$

Since x^* is periodic with period P, the first value of ξ for which Eq. (C8) is satisfied is for $P-\xi=\xi$, namely, $\xi=P/2$. Thus we have the result given in Eq. (C3).

Lemma 3

Based on the conditions and results of Lemmas 1 and 2, we arrive at Eq. (C1).

Proof

From Eq. (C3),

$$x^*[t; x^*[P/2; x^*(0)]] = x^*[t+P/2; x^*(0)]$$

$$= x^*[t; -x^*(0)] \quad \text{(C9)}$$

If Eq. (C9) is combined with the results of Eq. (C2), we arrive at Eq. (C1).

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